ATMAS Mathematics Specialist
Investigation 2.

2016

1 Introduction

Refer to the document "How to do a mathematical investigation" and the Investigation 2 Marking Rubric. (Investigation 2 will use a different rubric than investigation 1.)

Pay particular attention to presenting your mathematical thinking and working. This is *not* a research assignment where you look up somebody else's work; you must present your own thinking and your own mathematical ideas.

You must choose one of the following to investigate and present. The form of your presentation is up to you. The usual way would be to write up your investigation in a report, but you may choose to present your work as an ebook, a Powerpoint or Keynote presentation, a short film, etc. Note, however, that the communication marks are for clear communication, not for creativity and an innovative presentation will not be well received unless it is clear and appropriate for the purpose. You are not expected to present your findings to the class, but if you wish to do so time will be made available and your presentation may contribute towards the communication aspect of your mark.

2 Topics for investigation

2.1 Roots of Quadratic Equations (revisited)

Quadratic equations of the form $x^2 + bx + c = 0$ can have two real roots, one real root, or a pair of complex conjugate roots when b and c are real numbers. Investigate what happens when we allow b and c to be complex numbers.

2.2 Sun angle

The altitude of the sun at any given time is dependent on the time of day as well as season of the year.

The midday angle is greatest at the summer solstice and least at the winter solstice, with the amplitude being 23.4°. The variation of this daily maximum can be modelled by simple harmonic motion with a period of one year and mean altitude equal to $90° - \phi$ where ϕ is the latitude of the observer.

In addition to this annual variation there is the daily variation with amplitude equal to $90° - \phi$. This can also be modelled by simple harmonic motion with period of 24 hours.

Investigate how these two individual variations interact. Some of the things you might like to investigate include:

- If sunrise and sunset occur when sun altitude is zero, do these variations explain the change in length of day throughout the year?
- Is the model valid near the equator? Near the poles?
- At what time of year does the maximum daily sun altitude change the fastest from one day to the next?
- How could the model be used to design structures that give shade in summer and let the sun through in winter?
- Does the sun set fastest in summer, winter, autumn or spring?
- How well does the model match empirical measurements? (Note: never look directly at the sun; if you want to measure sun angle you should use a shadow stick or similar method.)

2.3 Atmospheric Pressure

Atmospheric pressure at height h above the surface of the Earth can be thought of as caused by the weight of the column of air above height h. For constant temperature, the density of air at a given height is proportional to pressure. This gives rise to a differential equation:

$$\frac{dP}{dh} = k_1 P$$

where P is pressure and h is height.

Use this to create a simple mathematical model of how air pressure varies with height above the surface of the Earth.

This simple model assumes constant temperature, but in reality temperature varies significantly. Below about 12km height temperature declines linearly (or near enough) from around 290K at the surface to around 210K at 12km. Density is inversely proportional to temperature, so we get a slightly more detailed model from the differential equation

$$\frac{dP}{dh} = k_2 \frac{P}{T}$$

where T is itself a function of height, as described.

Use this to create a more accurate mathematical model of how air pressure varies with height up to 12km.

2.4 Surface of Revolution

When curve $y = f(x)$ is rotated around the $x-$axis on the interval $a < x < b$, the surface area is given by

$$S = \int_a^b \left(2\pi f(x)\sqrt{1 + (f'(x))^2}\right) dx$$

Investigate how this can be used to obtain familiar formulas for surface area for shapes with curved surfaces such as cylinders and cones.

2.5 Lunar Lander

An old-school computer game sets the challenge of landing on the moon with a limited amount of fuel. The moon applies constant acceleration of 1.6ms^{-2} downwards while your rockets can accelerate you with force $f \propto \frac{dm}{dt}$ where $\frac{dm}{dt}$ represents how fast you burn your fuel and has an upper limit and the constant of proportionality is the rockets' exhaust velocity. The upward acceleration that results from this force is given by $a = \frac{f}{M}$ where M is the mass of your lander, including any fuel you haven't burnt yet.

When I played I liked to decelerate gradually from the start but I invariably ran out of fuel. Is it more fuel efficient to coast for as long as you can then burn hard when you get near the surface? Why?

Making some reasonable assumptions (such as assuming that you start with half the total mass as fuel, and that the maximum acceleration is around 20ms^{-2}, etc.) investigate different strategies. Is there an optimal approach?