How to do a mathematical investigation

Consider this problem:

> *I go into a locker bay that contains 150 (unlocked) lockers numbered consecutively 1 to 150. I go through and open every locker door. Then I go back to the beginning and starting at locker number 2, I close every second door. Now I go back and start at locker number 3, I visit every third locker, opening its door if it is closed, or closing it if it is open. Then starting at number 4, I visit every fourth locker in the same way, and so on until eventually on the last time round I start at locker number 150. How many lockers are now open?*

Rather than simply considering this as an example of mathematical problem solving, let us consider how we might further investigate this. I could tell you now that the answer is 12, but that's really not very interesting. Far more interesting is the process that allows us to find both the answer and make other discoveries along the way.

1. In the beginning

First, you have to understand what it is you are investigating. What is the problem? What are its essential features? What is unknown? What are the data? What conditions are specified? Is there enough information given? Is some of the information redundant? Contradictory?

Often at this stage it is necessary to draw a clear diagram. If you have not included a diagram, make sure you can justify why it would not be helpful.

Introduce suitable notation.

Identify and tease apart the separate parts of the problem or situation being investigated and list them.

What features of the problem might be changed to make the problem more general? What features could be restricted to find special cases?

For our sample problem, a suitable diagram might be something like this:

	Locker													
Turn	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**...**	**150**
1	○	○	○	○	○	○	○	○	○	○	○	○	○	...	○
2	○	●	○	●	○	●	○	●	○	●	○	●	○	...	●
3	○	●	●	●	○	○	○	●	●	●	○	○	○	...	○
4	○	●	●	○	○	○	○	○	●	●	○	●	○	...	○
5	○	●	●	○	●	○	○	○	●	○	○	●	○	...	●
6	○	●	●	○	●	●	○	○	●	○	○	○	○	...	○

Key	
Opened this turn	○
Closed this turn	●
Unchanged (open)	○
Unchanged (closed)	●

(There are many other ways you could draw a suitable diagram for this problem.)

The essential features include:
- There are 150 lockers
- Lockers are numbered sequentially with no gaps. (The no gaps is implied, but not explicit in the problem statement.)
- Whenever a locker is visited it is opened if it is closed, and closed if it is open.
- The first 'turn' opens all the lockers, so (based on the previous item) they all start closed.
- A 'turn' involves going through the whole locker bay and visiting lockers whose number is a multiple of the turn number.
- The problem involves 150 'turns'.

The 'unknown' to be found is the number of lockers open after 150 turns.

A more general version of the problem would involve determining how many lockers remained open for a total of n lockers (i.e. generalising from 150 lockers to n lockers).

A different more general version would be to be able to predict how many lockers are open after m turns.

No special cases are apparent.

Terminology: Let t be the turn number (where 'turn' is as described above). Let $L_{i,m}$ be the state of locker i after turn m, 1 if open, 0 if closed. Let O_m be the number of open lockers after turn m, i.e.

$$O_m = \sum_{i=1}^{150} L_{i,m}$$

2. Devising a plan

Have you seen something like this before? Have you encountered a similar situation? Do you know of a theorem that might be useful? Should you introduce an auxiliary element into the situation that would facilitate things? Is there a simpler version of the problem that could be used as a stepping stone? Can you restate the problem? Can you restate it still differently? These are all useful questions to ask in solving mathematical problems. In some senses, however, a mathematical investigation really begins by simply asking, "What do I know, and what else could I work out?" Rather than being totally goal directed, an investigation is an exploration of a problem or situation and anything that produces additional information is useful, whether it moves towards answering the problem's question or not. Because of this, when we say "devising a plan" you should not imagine that at this stage you need to see the whole path to the answer laid out before you, and indeed in some investigations there is no single "answer". In most cases, you will not be able to tell beforehand exactly which approaches will be useful, although the more practice you have with mathematical investigation the more reliable your intuition is likely to be.

For our sample problem a promising plan would be to examine simpler problems with fewer lockers, starting with $n = 1$ and increasing from there, in order to see if a pattern becomes apparent that could be extended to $n = 150$.

Another possible plan would be to calculate the number of lockers that change state each term, and use that to determine the number open after 150 turns, but this is problematic after the first few turn. It is easy to determine how many lockers change state, but much harder to determine how many go from open to closed, and how many from closed to open, In turn 5, for instance we can work out simply that 30 lockers will change state, but we cannot easily determine what the *net* change will be unless we model all 150 lockers through each turn. This would be time consuming to do manually, but would offer little difficulty to a spreadsheet-based solution. Technology based plans can be very useful, but make sure you make the necessary effort to truly *understand* the results; don't just copy results blindly.

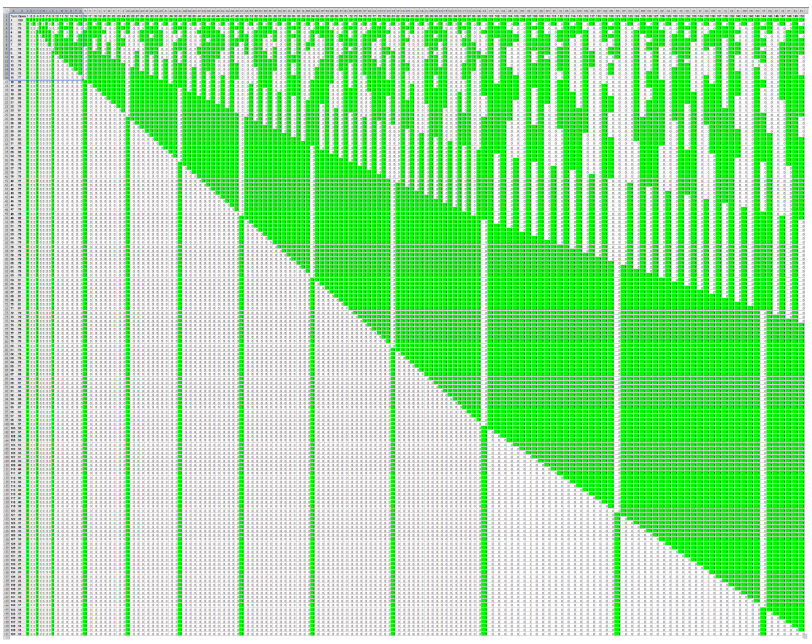
3. Carry Out the Plan

Carrying out your plan, check each step. Is it correct? How do you know? Can you prove it? What makes it correct? Do you know a theorem that supports it, or is there a new theorem that you can prove that flows from it? Although you may be working towards a particular solution, it is essential to keep your eyes open for other incidental insights along the way.

For our sample problem, the technology-based (spreadsheet) approach looks (in part) like this:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	Turn	Open	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1	150	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	2	75	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
4	3	75	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0
5	4	88	1	0	0	1	1	1	1	1	0	0	1	0	1	0	0
6	5	82	1	0	0	1	0	1	1	1	0	1	1	0	1	0	1
7	6	83	1	0	0	1	0	0	1	1	0	1	1	1	1	0	1
8	7	80	1	0	0	1	0	0	0	1	0	1	1	1	1	1	1
9	8	72	1	0	0	1	0	0	0	0	0	1	1	1	1	1	1
10	9	74	1	0	0	1	0	0	0	0	1	1	1	1	1	1	1
11	10	69	1	0	0	1	0	0	0	0	1	0	1	1	1	1	1
12	11	70	1	0	0	1	0	0	0	0	1	0	0	1	1	1	1
13	12	74	1	0	0	1	0	0	0	0	1	0	0	0	1	1	1
14	13	79	1	0	0	1	0	0	0	0	1	0	0	0	0	1	1
15	14	75	1	0	0	1	0	0	0	0	1	0	0	0	0	0	1
16	15	77	1	0	0	1	0	0	0	0	1	0	0	0	0	0	0

The formula in cell C2 is **=IF(MOD(C$1,$A3)=0,1-C2,C2)** and this formula has been copied right and then down to fill the whole spreadsheet up to column EV and row 151.



The only other formula is in column B. The formula in B2 is **=SUM(C2:EV2)** and this has been copied down to fill column B. Conditional formatting has been used to colour open lockers (1s) green to help spot at a glance which cells represent open lockers. There are some interesting patterns apparent from this small section but if we zoom out to see the whole thing we effectively get a graphical representation of the problem and some unexpected features jump out. We could, of course, at this stage scroll down and determine that after turn 150 only 12 lockers remain open, but this is in many ways the least interesting aspect of this spreadsheet, even though it answers the question we started with. So what can we identify here as needing further investigation?

- Apart from the vertical pinstripes, the whole bottom-left half of the spreadsheet represents closed lockers, and none of the lockers change state once they are in this bottom-left half. Why is that?
- This diagonal appears to have a slight bend about 2/3 of the way across. Why?
- Those pinstripes: what numbers do they represent? What is it about them that makes them the only lockers still open at the end?
- Why are the pinstripes closer together for smaller locker numbers?
- The (almost) solid section is bounded above by another diagonal line and all lockers in this region (apart from the pinstripes) are open.
- Above the solid green section there is a section of stripes where every second locker is open (except for the pinstripes which are the reverse of what would otherwise be the case).
- Above this stripy section there is another section with wider stripes: three lockers open, three closed, etc. (except for the pinstripes which are the reverse of what would otherwise be the case).
- Above this the situation looks more chaotic, but there still seems to be something happening with period 6, because there is an entirely white or mostly white (i.e. closed) column that runs down from about turn 3 to the stripy section every 6 lockers. What are these lockers that spend most or all of the early turns closed? What causes that?

As I hope you can see, although we have an answer to the original problem posed, there are still many opportunities for investigation. Some of this further investigation might make use of a smaller version of the problem, say 10 lockers, in order to try to understand what is happening and what causes the features observed.

4. Looking back

Can you check the result(s) you have obtained? Can you check the arguments or proofs you have developed? Can you arrive at the same results differently? Often once you realise what the answer is to a problem a more direct approach becomes obvious. The answer may reveal some aspect to the problem that was hidden earlier. (In your investigation you should be sure to discuss this, but resist the temptation to rewrite your investigation to focus on these previously hidden aspects rather than the approach you actually took.) What is the significance of what you've found? Are there other situations where your discoveries might be useful?

The nature of investigation is that the plan, do, reflect steps are iterative. As you identify other interesting things to investigate you begin another round of planning, doing and then checking and reflecting on what you've done. In some ways a good investigation never finishes, because the insights you gain will in turn raise other questions or suggest other related avenues of investigation. You might like to explore some of these related topics (as stated earlier above); others you should simply list as ideas for further investigation.

Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. ... A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted. (from *Solve It*, George Pólya, 1945, Princeton)

5. Communicating results

"It is impossible for a cube to be the sum of two cubes, a fourth power to be the sum of two fourth powers, or in general for any number that is a power greater than the second to be the sum of two like powers. I have discovered a truly marvellous demonstration of this proposition that this margin is too narrow to contain." Pierre de Fermat, handwritten note in the margin of *Arithmetica* by Diophantus.

You may discover truly marvellous things in the course of your investigation; don't keep them to yourself! Too often students fail to appreciate the importance of clearly communicating their work. You need to communicate *what* you chose to investigate, *how* you went about the investigation, any *discoveries* you made, what your *conclusions* are in terms of the significance of your discoveries, and the *reasoning, proof or justification* for your conclusions.

The most common way to communicate an investigation is by writing up an investigation report, but other approaches are also acceptable. You may also have to present your work to others, in which case a more visual presentation might be appropriate. Some students may wish to make a movie detailing their results, or make a Prezi presentation, or put together web pages utilising a variety of different media. All are acceptable provided they successfully document every aspect of the investigation.

What a mathematical investing is not ...

Finally, some things to avoid.

A mathematical investigation is not just about the things that worked.

You will often follow up a lot of false leads when following up your investigation. Make sure you keep a record of them. The things that didn't work are sometimes just as valuable as those that did in terms of the insight you gain. Even if for no other reason, keeping records of failed attempts helps ensure you don't simply repeat them. It's not necessary to write up every single failed attempt, but when your work is being marked, your teacher will look for evidence of other attempts. If a failed attempt gives you insight into *why* it failed, you should include that in your write-up or presentation. (If you were able to strike a successful approach on your first attempt, please don't make up fake failed attempts to look complete ... just make a point of stating that you got where you did on the first attempt, and what it was that you observed that led you straight to a productive approach.)

A mathematical investigation is not the same as a copying and pasting the answers from an Internet search.

As part of your investigation you will need to develop your own ideas and a deep understanding of the subject you are investigating. You can expect to be asked questions about your work and it will become obvious if you do not truly understand the work you are submitting.

A mathematical investigation is not getting your tutor to do all the hard thinking.

If you have a tutor, this can be tricky. People who enjoy mathematics find a mathematical investigation quite seductive and it can be difficult for your tutor to know when to stop. The same, indeed, is true of your teacher or other teachers at Maths Club. It's ok for someone to give you hints or ask you leading questions, but don't let them rob you of the "Aha!" moments that you only get by arriving at insights yourself. Make sure you know how to politely say, "Thank you, I think I'll try to take it from there."